THE R-MATRIX OF QUANTUM DOUBLES OF NICHOLS ALGEBRAS OF DIAGONAL TYPE

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Abstract. Let H be the quantum double of a Nichols algebra of diagonal type. We compute the R-matrix of 3-uples of modules for general finite-dimensional highest weight modules over H.

We calculate also a multiplicative formula for the universal R-matrix when H is finite dimensional.

1. Introduction

A remarkable property of quantum groups, introduced by Drinfeld and Jimbo in the eighties, is the existence of an R- matrix for their categories of modules. This R-matrix is related with the existence of solutions of the Yang-Baxter equation. An explicit formula for the universal R-matrix of quantum groups was obtained in the nineties [KR, LS, Ro1], and extended to quantized enveloping superalgebras [KhT, Y] of finite-dimensional Lie superalgebras.

We can deduce the existence of this R-matrix for quantized enveloping (super)algebras because they can be obtained as quotients of quantum doubles of bosonizations of the positive part by group algebras, and these quantum doubles are quasi-triangular.

A natural generalization of the positive part of these quantized enveloping (super)algebras are the Nichols algebras of diagonal type [AS]. They admit a root system and a Weyl groupoid [HY1, HS] controlling the structure of these algebras. Moreover, the classification of these Nichols algebras with finite root system includes (properly) the positive part of quantized enveloping algebras of finite dimensional contragradient Lie superalgebras and simple Lie algebras. It is natural then to ask for a formula of the R-matrix in this general context. We answer this question for the subfamily of finite-dimensional representations with a highest weight in a general context, and obtain an explicit formula for the universal R-matrix when the Nichols algebra is finite-dimensional.

The organization of the paper is as follows. In Section 2 we recall definitions and results needed for our work. They are related with quantum doubles and properties of Nichols algebras of diagonal type. We stress the

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importance of the Weyl groupoid and the generalized version of root systems. In Section 3 we work over arbitrary Nichols algebras of diagonal type and compute the R-matrix of 3-uples of finite-dimensional modules, generalizing the results in [T]. We restrict our attention to highest weight modules, which give maybe the most important subfamily of representations. Finally in Section 4 we compute the universal R-matrix for quantum doubles of finite-dimensional Nichols algebras. The formula involves the multiplication of quantum exponentials of root vector powers, generalizing the classical ones for quantum groups.

Notation. We denote by $\mathbb{N}$ the set of natural numbers, and by $\mathbb{N}_0$ the set of non-negative integers.

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. All the vector spaces, algebras and tensor products are over $\mathbf{k}$. We shall use the usual notation for q-combinatorial numbers: for each $q \in \mathbf{k}^\times$, $n \in \mathbb{N}$, $0 \le k \le n$,

$$(n)_q = 1 + q + \ldots + q^{n-1}, \qquad (n)_q! = (1)_q(2)_q \cdots (n)_q,$$
$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!}.$$

Let A be an associative algebra. Given an element $a \in A$ such that $a^N = 0$, we define the *q-exponential*, for each q which is not a root of unity, or it is a root of unity of order $\ge N$:

$$\exp_q(a) = \sum_{i=0}^{N-1} \frac{a^i}{(i)_q!}. \tag{1.1}$$

Let $\theta \in \mathbb{N}$. $\{\alpha_i\}_{1\le i\le\theta}$ will denote the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^\theta$. Given an $\mathbb{Z}$-linear automorphism $s : \mathbb{Z}^\theta \to \mathbb{Z}^\theta$ and a bicharacter $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$, $s^*\chi$ will denote the bicharacter

$$(s^*\chi)(\alpha,\beta) := \chi\left(s^{-1}(\alpha), s^{-1}(\beta)\right), \qquad \alpha,\beta \in \mathbb{Z}^\theta. \tag{1.2}$$

Given a Hopf algebra H with coproduct Δ, we will use the classical Sweedler notation $\Delta(h) = h_1 \otimes h_2$, $h \in H$, and denote

$$\Delta^{(2)} := (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

Given $R = \sum_i a_i \otimes b_i \in H \otimes H$, we set the following elements of $H \otimes H \otimes H$:

$$R^{(1,2)} = \sum_i a_i \otimes b_i \otimes 1, \quad R^{(1,3)} = \sum_i a_i \otimes 1 \otimes b_i, \quad R^{(2,3)} = \sum_i 1 \otimes a_i \otimes b_i.$$

Recall that a (right) coideal subalgebra of H is a subalgebra A of H such that $\Delta(A) \subseteq H \otimes A$.

2. Preliminaries

We recall some definitions and results which will be useful in the rest of this work. They are mainly related with quantum doubles of Hopf algebras and Nichols algebras of diagonal type.

2.1. **Skew-Hopf pairings and R-matrices.** Let A, B be two Hopf algebras. A *skew Hopf pairing* between A and B (see [J, Section 3.2.1], [KS, Section 8.2]) is a linear map $\eta : A \otimes B \to \mathbf{k}$ such that

$$\begin{aligned} &\eta(xx', y) = \eta(x', y_1)(x, y_2), && \eta(x, 1) = \varepsilon(x), \\ &\eta(x, yy') = \eta(x_1, y)(x_2, y'), && \eta(1, y) = \varepsilon(y), \\ &\eta(\mathcal{S}(x), y) = \eta(x, \mathcal{S}^{-1}(y)), \end{aligned}$$

for all $x, x' \in A$, $y, y' \in B$. In such case, $A \otimes B$ admits a unique structure of Hopf algebra, denoted by $\mathcal{D}(A, B, \eta)$ and called the *quantum double* associated to η, such that the morphisms $A \to A \otimes B$, $a \mapsto a \otimes 1$, $B \to A \otimes B$, $b \mapsto 1 \otimes b$ are Hopf algebra morphisms and

$$(a \otimes 1)(1 \otimes b) = a \otimes b, \qquad (1 \otimes b)(a \otimes 1) = \eta\left(a_1, \mathcal{S}(b_1)\right)(a_2 \otimes b_2)\eta(a_3, b_3).$$

When A is finite-dimensional and η is not degenerate, B is identified with the Hopf algebra A^*. $\mathcal{D}(A, B, \eta) = \mathcal{D}(A)$ is the *Drinfeld double* of A, which admits an R-matrix:

$$\mathcal{R} := \sum_{i \in I} (1 \otimes b_i) \otimes (a_i \otimes 1), \tag{2.1}$$

where $\{a_i\}_{i \in I}$, $\{b_i\}_{i \in I}$ are dual bases of A, B: $\eta(a_i, b_j) = \delta_{ij}$.

2.2. **Weyl groupoids and convex orders on finite root systems.** We recall the definitions of Weyl groupoid and generalized root system following [CH]. Fix a set $\mathcal{X} \neq \emptyset$ and a finite set I. Set also for each $i \in I$ a map $r_i : \mathcal{X} \to \mathcal{X}$, and for each $X \in \mathcal{X}$ a generalized Cartan matrix $A^X = (a_{ij}^X)_{i,j \in I}$ in the sense of [Ka].

Definition 2.1. [CH, HY1] The quadruple $\mathcal{C} := \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{C}})$ is a *Cartan scheme* if $r_i^2 = \mathrm{id}$ for all $i \in I$, and $a_{ij}^X = a_{ij}^{r_i(X)}$ for all $X \in \mathcal{X}$ and $i, j \in I$. For each $i \in I$ and each $X \in \mathcal{X}$ set s_i^X as the $\mathbb{Z}$-linear automorphism of $\mathbb{Z}^I$ determined by

$$s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i, \qquad j \in I. \tag{2.2}$$

The *Weyl groupoid* of $\mathcal{C}$ is the groupoid $\mathcal{W}(\mathcal{C})$ whose set of objects is $\mathcal{X}$ and whose morphisms are generated by s_i^X; here we consider s_i^X as an element in $\mathrm{Hom}(X, r_i(X))$, $i \in I$, $X \in \mathcal{X}$.

Given a Cartan scheme $\mathcal{C}$, and for each $X \in \mathcal{X}$ a set $\Delta^X \subset \mathbb{Z}^I$, we say that $\mathcal{R} := \mathcal{R}(\mathcal{C}, (\Delta^X)_{X \in \mathcal{X}})$ is a *root system of type* $\mathcal{C}$ if

- for all $X \in \mathcal{X}$, $\Delta^X = (\Delta^X \cap \mathbb{N}_0^I) \cup -(\Delta^X \cap \mathbb{N}_0^I)$. We call $\Delta_+^X := \Delta^X \subset \mathbb{N}_0^I$ the set of *positive roots*, and $\Delta_-^X := -\Delta_+^X$ the set of *negative roots*.

- for all $i \in I$ and all $X \in \mathcal{X}$, $\Delta^X \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}$.
- for all $i \in I$ and all $X \in \mathcal{X}$, $s_i^X(\Delta^X) = \Delta^{r_i(X)}$.
- set $m_{ij}^X := |\Delta^X \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$; then $(r_ir_j)^{m_{ij}^X}(X) = (X)$ for all $i \neq j \in I$ and all $X \in \mathcal{X}$.

We assume that $\mathcal{W}(\mathcal{C})$ is a connected groupoid: $\operatorname{Hom}(Y, X) \neq \emptyset$, for all $X, Y \in \mathcal{X}$. For any $X \in \mathcal{X}$, let $\operatorname{Hom}(\mathcal{W}, X) := \cup_{Y\in\mathcal{X}} \operatorname{Hom}(Y, X)$, and $\Delta^{X\ re} := \{w(\alpha_i) : \ i \in I, \ w \in \operatorname{Hom}(\mathcal{W}, X)\}$, the set of *real roots* of X. Clearly, $\Delta^{X\ re} \subseteq \Delta^X$, and $w(\Delta^X) = \Delta^Y$ for any $w \in \operatorname{Hom}(Y, X)$. We say that $\mathcal{R}$ is *finite* if Δ^X is finite for some $X \in \mathcal{X}$.

Note that each $w \in \operatorname{Hom}(\mathcal{W}, X_1)$ is a product $s_{i_1}^{X_1} s_{i_2}^{X_2} \cdots s_{i_m}^{X_m}$, where $X_j = r_{i_{j-1}} \cdots r_{i_1}(X_1)$, $i \geq 2$; we fix the notation $w = \mathrm{id}_{X_1} s_{i_1} \cdots s_{i_m}$ to mean that $w \in \operatorname{Hom}(\mathcal{W}, X_1)$, because the elements $X_j \in \mathcal{X}$ are determined by the previous condition. The *length* of w is

$$\ell(w) = \min\{n \in \mathbb{N}_0 : \ \exists i_1, \dots, i_n \in I \text{ such that } w = \mathrm{id}_X s_{i_1} \cdots s_{i_n}\}.$$

Proposition 2.2. [CH, Prop. 2.12] *Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$, $\ell(w) = m$. The roots $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$ are positive and pairwise different.*

Moreover, if $\mathcal{R}$ is finite and w is an element of maximal length, then $\{\beta_j\} = \Delta_+^X$, so all the roots are real. □

For the last part of this subsection, assume that $\mathcal{R}$ is finite.

Definition 2.3. [A2] Given a root system $\mathcal{R}$ and a fixed total order $<$ on Δ_+^X, we say that it is *convex* if for each $\alpha, \beta \in \Delta_+^X$ such that $\alpha < \beta$ and $\alpha + \beta \in \Delta_+^X$, then $\alpha < \alpha + \beta < \beta$. It is said *strongly convex* if for each ordered subset $\alpha_1 \leq \dots \leq \alpha_k$ of elements of Δ_+^X such that $\alpha := \sum \alpha_i \in \Delta_+^X$, it holds that $\alpha_1 < \alpha < \alpha_k$.

Theorem 2.4. [A2] *Given an order on Δ_+^X, the following are equivalent:*

1. *the order is convex,*
2. *the order is strongly convex,*
3. *the order is associated with a reduced expression of the longest element.* □

2.3. **Weyl groupoid of a Nichols algebra of diagonal type.** For each bicharacter $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$, set $q_{ij}(\chi) = \chi(\alpha_i, \alpha_j)$. Given $1 \leq i \leq \theta$, we say that χ is *i-finite* if for all $1 \leq j \neq i \leq \theta$ there exists $m \in \mathbb{N}_0$ such that $(m+1)_{q_{ii}}(1 - q_{ii}^2 q_{ij} q_{ji}) = 0$. In such case, define

$$a_{ii}^\chi = 2, \qquad a_{ij}^\chi = -\min\{m \in \mathbb{N}_0 | (m+1)_{q_{ii}}(1 - q_{ii}^2 q_{ji} q_{ij}) = 0\},$$

and set s_i^χ as the $\mathbb{Z}$-linear automorphism of $\mathbb{Z}^\theta$ given by (2.2). If χ is i-finite for all i, $A^\chi = (a_{ij}^\chi)_{1\leq i,j\leq\theta}$ is the *generalized Cartan matrix* associated to χ.

Let $\mathcal{X}$ be the set of all the bicharacters of $\mathbb{Z}^\theta$. We define $r_i : \mathcal{X} \to \mathcal{X}$ by $r_i(\chi) = (s_i^\chi)^* \chi$ if χ is i-finite, or $r_i(\chi) = \chi$ otherwise. Such r_i's are involutions and $\mathcal{G}(\chi)$ will denote the orbit of χ by the action of the group of bijections generated by the r_i's.

Note that $\mathcal{C}(\chi) = \mathcal{C}\left(\{1,\dots,\theta\}, \mathcal{G}(\chi), (r_i)_{1\le i\le\theta}, (C^v)_{v\in\mathcal{G}(\chi)}\right)$ is a connected Cartan scheme, see [HY1, HY2]. Therefore the associated Weyl groupoid $\mathcal{W}(\chi)$ is called the *Weyl groupoid of* χ.

There exists a close relation between the root system and the set $\mathcal{K}(V)$ of graded coideal subalgebras, as it is stated in [HS]. We refer the reader to this reference for the definitions of the coideal subalgebra $B(w)$ and the Duflo order.

Theorem 2.5. [HS] *For each* $w \in \operatorname{Hom}(\mathcal{W}, V)$ *there exists a unique right coideal subalgebra* $B(w) \in \mathcal{K}(V)$ *such that its Hilbert series is*

$$\mathcal{H}_{B(w)} = \prod_{\beta\in\Lambda_+^V(w)} \mathbf{q}_{N_\beta}(X^\beta). \tag{2.3}$$

Moreover, the correspondence $w \mapsto B(w)$ *gives an order preserving and order reflecting bijection between* $\operatorname{Hom}(\mathcal{W}, V)$ *and* $\mathcal{K}(V)$*, where we consider the Duflo order over* $\operatorname{Hom}(\mathcal{W}, V)$ *and the inclusion order over* $\mathcal{K}(V)$*; i.e.* $w_1 \le_D w_2$ *if and only if* $B(w_1) \subset B(w_2)$. $\square$

2.4. **Lusztig Isomorphisms of Nichols algebras.** Set χ, (q_{ij}) as in Subsection 2.3. $\mathcal{U}(\chi)$ will denote the algebra presented by generators E_i, F_i, K_i, K_i^{-1}, L_i, L_i^{-1}, $1 \le i \le \theta$, and relations

$$\begin{aligned}
XY &= YX, & X, Y &\in \{K_i^{\pm1}, L_i^{\pm1} : 1 \le i \le \theta\},\\
K_iK_i^{-1} &= L_iL_i^{-1} = 1, & E_iF_j - F_jE_i &= \delta_{i,j}(K_i - L_i)\\
K_iE_jK_i^{-1} &= q_{ij}E_j, & L_iE_jL_i^{-1} &= q_{ji}^{-1}E_j,\\
L_iF_jL_i^{-1} &= q_{ji}F_j, & K_iF_jK_i^{-1} &= q_{ij}^{-1}F_j.
\end{aligned}$$

$\mathcal{U}^{+0}(\chi)$ (respectively, $\mathcal{U}^{-0}(\chi)$) will denote the subalgebra generated by K_i, K_i^{-1} (respectively, L_i, L_i^{-1}), $1 \le i \le \theta$, and $\mathcal{U}^0(\chi)$ will denote the subalgebra generated by K_i, K_i^{-1}, L_i and L_i^{-1}. Also, $\mathcal{U}^+(\chi)$ (respectively, $\mathcal{U}^-(\chi)$) will denote the subalgebra generated by E_i (respectively, F_i), $1 \le i \le \theta$.

$\mathcal{U}(\chi)$ is a $\mathbb{Z}^\theta$-graded Hopf algebra, with graduation determined by the following conditions:

$$\deg(K_i) = \deg(L_i) = 0, \qquad \deg(E_i) = \alpha_i, \qquad \deg(F_i) = -\alpha_i.$$

$\mathcal{U}(\chi)$ admits a Hopf algebra structure, with comultiplication determined by

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, & \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i,\\
\Delta(L_i) &= L_i \otimes L_i, & \Delta(F_i) &= F_i \otimes L_i + 1 \otimes F_i,
\end{aligned}$$

and then $\varepsilon(K_i) = \varepsilon(L_i) = 1$, $\varepsilon(E_i) = \varepsilon(F_i) = 0$.

Note that $\mathcal{U}^0(\chi)$ is isomorphic to $\mathbf{k}\mathbb{Z}^{2\theta}$ as Hopf algebras, and the subalgebra $\mathcal{U}^{\ge0}(\chi)$ (respectively, $\mathcal{U}^{\le0}(\chi)$) generated by $\mathcal{U}^+(\chi)$, $K_i^{\pm1}$, $1 \le i \le \theta$, (respectively, $\mathcal{U}^-(\chi)$, $L_i^{\pm1}$) is isomorphic to $T(V)\#\mathbf{k}\mathbb{Z}^\theta$ (respectively, $T(V^*)\#\mathbf{k}\mathbb{Z}^\theta$). $\mathcal{U}(\chi)$ is the associated quantum double.

Here, $\mathcal{U}^+(\chi)$ is isomorphic to $T(V)$ as braided graded Hopf algebras in the category of Yetter-Drinfeld modules over $\mathbf{k}\mathbb{Z}^\theta$, with actions and coactions given by:

$$K_i \cdot E_j = q_{ij} E_j, \qquad \delta(E_i) = K_i \otimes E_i,$$

and similar equations for F_j, L_i. $\underline{\Delta}$ will denote the braided comultiplication of $\mathcal{U}^+(\chi)$. As it is $\mathbb{N}_0$-graded, we will consider $\underline{\Delta}_{n-k,k}(E)$, the component of $\underline{\Delta}(E)$ in $\mathcal{U}^+(\chi)_{n-k} \otimes \mathcal{U}^+(\chi)_k$, for each $E \in \mathcal{U}^+(\chi)$ homogeneous of degree n, and $k \in \{0, 1, \dots, n\}$.

By [H2, Prop. 4.14], the multiplication $m : \mathcal{U}^+(\chi) \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^-(\chi) \to \mathcal{U}(\chi)$ is an isomorphism of $\mathbb{Z}^\theta$-graded vector spaces.

We consider some isomorphisms involving $\mathcal{U}(\chi)$ [H2, Section 4.1].

(a) Let $\underline{a} = (a_1, \dots, a_\theta) \in (\mathbf{k}^\times)^\theta$. There exists a unique algebra automorphism $\varphi_{\underline{a}}$ of $\mathcal{U}(\chi)$ such that

$$\varphi_{\underline{a}}(K_i) = K_i, \quad \varphi_{\underline{a}}(L_i) = L_i, \quad \varphi_{\underline{a}}(E_i) = a_i E_i, \quad \varphi_{\underline{a}}(F_i) = a_i^{-1} F_i. \tag{2.4}$$

(b) There exists a unique algebra antiautomorphism Ω of $\mathcal{U}(\chi)$ such that

$$\Omega(K_i) = K_i, \quad \Omega(L_i) = L_i, \quad \Omega(E_i) = F_i, \quad \Omega(F_i) = E_i. \tag{2.5}$$

It satisfies the relation $\Omega^2 = \mathrm{id}$.

As in [HY2, H2], $\mathcal{I}^+(\chi)$ will denote the ideal of $\mathcal{U}^+(\chi)$ such that the quotient $\mathcal{U}^+(\chi)/\mathcal{I}^+(\chi)$ is isomorphic to the Nichols algebra of V; that is, the greatest braided Hopf ideal of $\mathcal{U}^+(\chi)$ generated by elements of degree ≥ 2. Set $\mathcal{I}^-(\chi) = \Omega(I^+(\chi))$, where ϕ_4 is the anti-automorphism of algebras determined by (2.5), and

$$\mathfrak{u}^\pm(\chi) := \mathcal{U}^\pm(\chi)/\mathcal{I}^\pm(\chi), \qquad \mathfrak{u}(\chi) := \mathcal{U}(\chi)/(\mathcal{I}^-(\chi) + \mathcal{I}^+(\chi)),$$

and $\mathfrak{u}^{\geq 0}(\chi)$, $\mathfrak{u}^{\leq 0}(\chi)$ the corresponding images on the quotient. Note that $\mathfrak{u}(\chi)$ is the quantum double of $\mathfrak{u}^+(\chi)\#\mathbf{k}\mathbb{Z}^\theta$. The following result follows by [H2, Lemma 6.5, Theorem 6.12].

Proposition 2.6. [HY2, Proposition 3.5], [H2, Theorem 5.8] *There exists a unique non-degenerate skew-Hopf pairing $\eta : \mathfrak{u}^+(\chi) \otimes \mathfrak{u}^-(\chi)$ such that*

$$\eta(K_i, L_j) = q_{ij}, \qquad \eta(E_i, F_j) = -\delta_{ij}, \qquad \eta(E_i, L_j) = \eta(K_i, F_j) = 0. \tag{2.6}$$

for all $1 \leq i, j \leq \theta$. It satisfies the following condition: for all $E \in \mathfrak{u}^+(\chi)$, $F \in \mathfrak{u}^-(\chi)$, $K \in \mathfrak{u}^{+0}(\chi)$, $L \in \mathfrak{u}^{-0}(\chi)$

$$\eta(EK, FL) = \eta(E, F)\eta(K, L). \tag{2.7}$$

Moreover, if $\beta \neq \gamma \in \mathbb{N}_0^\theta$, then $\eta|_{\mathfrak{u}^+(\chi)_\beta \otimes \mathfrak{u}^-(\chi)_{-\gamma}} \cong 0$. $\square$

Assume that all the integers a_{ij}^v are defined, $v \in \mathcal{G}(\chi)$, so the automorphisms s_p^v are defined. For simplicity, we denote $\underline{E}_i$, $\underline{F}_i$, $\underline{K}_i$, $\underline{L}_i$ the generators corresponding to $\mathcal{U}(s_p^*\chi)$, $a_{ij} = a_{ij}^\chi$, $q_{ij} = q_{ij}^\chi$, $\underline{q}_{ij} = q_{ij}^{s_p^*\chi}$. We define also the scalars

$$\lambda_i(\chi) := (-a_{pi})_{q_{pp}} \prod_{s=0}^{-a_{pi}-1} (q_{pp}^s q_{pi} q_{ip} - 1), \qquad i \neq p. \tag{2.8}$$

Fix $p \in \{1, \dots, \theta\}$. If $i \neq p$ we consider the elements [H2],

$$E_{i,0(p)}^+, E_{i,0(p)}^- := E_i, \qquad F_{i,0(p)}^+, F_{i,0(p)}^- := F_i,$$

and recursively,

$$\begin{aligned} E_{i,m+1(p)}^+ &:= E_p E_{i,m(p)}^+ - (K_p \cdot E_{i,m(p)}^+) E_p = (\operatorname{ad}_c E_p)^{m+1} E_i, \\ E_{i,m+1(p)}^- &:= E_p E_{i,m(p)}^- - (L_p \cdot E_{i,m(p)}^-) E_p, \\ F_{i,m+1(p)}^+ &:= F_p F_{i,m(p)}^+ - (L_p \cdot F_{i,m(p)}^+) F_p, \\ F_{i,m+1(p)}^- &:= F_p F_{i,m(p)}^- - (K_p \cdot F_{i,m(p)}^-) F_p. \end{aligned}$$

If p is explicit, we simply denote $E_{i,m(p)}^\pm$ by $E_{i,m}^\pm$. By [H2, Corollary 5.4],

$$E_{i,m}^+ F_i - F_i E_{i,m}^+ = (m)_{q_{pp}} (q_{pp}^{m-1} q_{pi} q_{ip} - 1) L_p E_{i,m-1}^+. \tag{2.9}$$

Theorem 2.7. *There exist algebra morphisms*

$$T_p, T_p^- : \mathfrak{u}(\chi) \to \mathfrak{u}(s_p^*\chi) \tag{2.10}$$

univocally determined by the following conditions:

$$\begin{aligned} T_p(K_p) &= T_p^-(K_p) = \underline{K}_p^{-1}, & T_p(K_i) &= T_p^-(K_i) = \underline{K}_p^{m_{pi}} \underline{K}_i, \\ T_p(L_p) &= T_p^-(L_p) = \underline{L}_p^{-1}, & T_p(L_i) &= T_p^-(L_i) = \underline{L}_p^{m_{pi}} \underline{L}_i, \\ T_p(E_p) &= \underline{F}_p \underline{L}_p^{-1}, & T_p(E_i) &= \underline{E}_{i,m_{pi}}^+, \\ T_p(F_p) &= \underline{K}_p^{-1} \underline{E}_p, & T_p(F_i) &= \lambda_p(s_p^*\chi)^{-1} \underline{F}_{i,m_{pi}}^+, \\ T_p^-(E_p) &= \underline{K}_p^{-1} \underline{F}_p, & T_p^-(E_i) &= \lambda_p(s_p^*\chi^{-1})^{-1} \underline{E}_{i,m_{pi}}^-, \\ T_p^-(F_p) &= \underline{E}_p \underline{L}_p^{-1}, & T_p^-(F_i) &= \underline{F}_{i,m_{pi}}^-. \end{aligned}$$

for every $i \neq p$. Moreover, $T_p T_p^- = T_p^- T_p = \mathrm{id}$, and there exists $\mu \in (\mathbf{k}^\times)^\theta$ such that

$$T_p \circ \phi_4 = \phi_4 \circ T_p^- \circ \varphi_\mu. \tag{2.11}$$

□

By [HY2, Proposition 4.2], we have for all $\alpha \in \mathbb{Z}^\theta$

$$T_p\big(\mathfrak{u}(\chi)_\alpha\big) = \mathfrak{u}(s_p^*\chi)_{s_p^\chi(\alpha)}. \tag{2.12}$$

3. R-matrix from a version of a universal R-matrix

Most of the ideas we shall give in this section are modifications of [T, Section 4]. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ be any bicharacter. We will compute an R-matrix for some modules of $\mathfrak{u}(\chi)$ from canonical elements of $\mathfrak{u}(\chi)$. If $M = |\Delta_+^\chi| < \infty$, the canonical elements can be obtained by Proposition 4.6.

3.1. **Equations for canonical elements.** We recall [HY2, (3.18), (3.19)]:

$$YX = \eta(X_1, \mathcal{S}(Y_1))\eta(X_3, Y_3)X_2Y_2, \tag{3.1}$$

$$XY = \eta(X_1, Y_1)\eta(X_3, \mathcal{S}(Y_3))Y_2X_2, \qquad X \in \mathfrak{u}^{\geq 0}(\chi),\ Y \in \mathfrak{u}^{\leq 0}(\chi). \tag{3.2}$$

Define the $\mathbf{k}$-linear homomorphism $\tau : \mathfrak{u}(\chi) \otimes \mathfrak{u}(\chi) \to \mathfrak{u}(\chi) \otimes \mathfrak{u}(\chi)$ by

$$\tau(X \otimes Y) := Y \otimes X.$$

Given $\mathbf{X} \in \mathfrak{u}^{\geq 0}(\chi)$, $\mathbf{Y} \in \mathfrak{u}^{\leq 0}(\chi)$, we define the $\mathbf{k}$-linear homomorphisms

$$\begin{aligned} &\hat{\eta}^{\leq}_{\mathbf{X}} : \mathfrak{u}^{\leq 0}(\chi) \to \mathbf{k}, && \hat{\eta}^{\leq}_{\mathbf{X}}(Y) := \eta(\mathbf{X}, Y), && Y \in \mathfrak{u}^{\leq 0}(\chi), \\ &\hat{\eta}^{\geq}_{\mathbf{Y}} : \mathfrak{u}^{\geq 0}(\chi) \to \mathbf{k}, && \hat{\eta}^{\geq}_{\mathbf{Y}}(X) := \eta(X, \mathbf{Y}), && X \in \mathfrak{u}^{\geq 0}(\chi). \end{aligned}$$

Lemma 3.1. *Let $1 \leq i \leq \theta$ and $\beta \in \mathbb{N}_0^\theta$. Set*

$$\mathbb{N}_0^\theta(\beta; i) := \{\, \gamma \in \mathbb{N}_0^\theta\ - \{0, \alpha_i\}, |\, \beta - \gamma \in \mathbb{N}_0^\theta - \{0, \alpha_i\}\}.$$

(i) *Let $\beta \notin \{0, \alpha_i, 2\alpha_i\}$, $Y \in \mathfrak{u}^-(\chi)_{-\beta}$. Set Y', $Y'' \in \mathfrak{u}^-(\chi)_{-\beta+\alpha_i}$ such that $[E_i, Y] = K_iY' - Y''L_i$. Then*

$$\begin{aligned} \Delta(Y) - (Y \otimes L^\beta + 1 \otimes Y + F_i \otimes Y''L^{\alpha_i} + Y' \otimes F_iL^{\beta-\alpha_i}) \\ \in \oplus_{\gamma \in \mathbb{N}_0^\theta(\beta;i)} \mathfrak{u}^-(\chi)_{-\gamma} \otimes \mathfrak{u}^-(\chi)_{-\beta+\gamma} L^\gamma. \end{aligned} \tag{3.3}$$

In particular,

$$(\hat{\eta}^{\leq}_{E_i} \otimes \mathrm{id})(\Delta(Y)) = -Y''L_i, \qquad (\mathrm{id} \otimes \hat{\eta}^{\leq}_{E_i})(\Delta(Y)) = -Y'. \tag{3.4}$$

(ii) *Let $\beta \notin \{0, \alpha_i, 2\alpha_i\}$, $X \in \mathfrak{u}^+(\chi)_\beta$. Set $X', X'' \in \mathfrak{u}^+(\chi)_{\beta-\alpha_i}$ such that $[X, F_i] = X''K_i - L_iX'$. Then*

$$\begin{aligned} \Delta(X) - (X \otimes 1 + K^\beta \otimes X + X''K^{\alpha_i} \otimes E_i + E_iK^{\beta-\alpha_i} \otimes X') \\ \in \oplus_{\gamma \in \mathbb{N}_0^\theta(\beta;i)} \mathfrak{u}^+(\chi)_\gamma K^{\beta-\gamma} \otimes \mathfrak{u}^+(\chi)_{\beta-\gamma}. \end{aligned} \tag{3.5}$$

In particular,

$$(\mathrm{id} \otimes \hat{\eta}^{\geq}_{F_i})(\Delta(X)) = -X''K_i, \qquad (\hat{\eta}^{\geq}_{F_i} \otimes \mathrm{id})(\Delta(X)) = -X'. \tag{3.6}$$

Proof. We prove ***(i)*** ; ***(ii)*** can be proved analogously. Note that

$$\Delta^{(2)}(E_i) = E_i \otimes 1 \otimes 1 + K_i \otimes E_i \otimes 1 + K_i \otimes K_i \otimes E_i.$$

Define $\bar{Y}'$, $\bar{Y}''$ as the elements of $\mathfrak{u}^-(\chi)_{-\beta+\alpha_i}$ satisfying the same property as (3.3) with $\bar{Y}'$, $\bar{Y}''$ in place of Y', Y''. By (3.1), we have

$$\begin{aligned} YE_i =& \eta(E_i, S(F_i))\eta(1, L^\beta)\bar{Y}''L^{\alpha_i} + \eta(K_i, S(1))\eta(1, L^\beta)E_iY \\ & + \eta(K_i, S(1))\eta(E_i, F_iL^{\beta-\alpha_i})K_i\bar{Y}' \\ =& \eta(E_i, -F_iL^{-\alpha_i})\eta(1, L^\beta)\bar{Y}''L^{\alpha_i} + \eta(K_i, 1)\eta(1, L^\beta)E_iY \\ & + \eta(K_i, 1)\eta(E_i, F_iL^{\beta-\alpha_i})K_i\bar{Y}' \\ =& \bar{Y}''L_i + E_iY - K_i\bar{Y}', \end{aligned}$$

so the proof is complete. □

Fix $\beta \in \mathbb{N}_0^\theta$ and $m_\beta := \dim \mathfrak{u}^+(\chi)_\beta = \dim \mathfrak{u}^-(\chi)_{-\beta}$. Fix also $\{E_x^{(\beta)}\}$, $\{F_y^{(\beta)}\}$ bases of the spaces $\mathfrak{u}^+(\chi)_\beta$, $\mathfrak{u}^-(\chi)_{-\beta}$, which are dual for η. The matrix $[\eta(E_x^{(\beta)}, F_y^{(\beta)})]_{1\le x,y\le m_\beta}$ is invertible, we call $[b_{xy}^{(\beta)}]_{1\le x,y\le m_\beta}$ to its inverse.

Lemma 3.2. *For all $X \in \mathfrak{u}^+(\chi)_\beta$, $Y \in \mathfrak{u}^-(\chi)_{-\beta}$ it holds:*

$$X = \sum_{x,y} b_{yx}^{(\beta)}\eta(X, F_y^{(\beta)})E_x^{(\beta)} \tag{3.7}$$

$$X = \sum_{x,y} b_{yx}^{(\beta)}\eta(E_x^{(\beta)}, Y)F_y^{(\beta)}. \tag{3.8}$$

Proof. We prove (3.7); the proof of (3.8) is similar. We have

$$\begin{aligned} \eta\left(\sum_{x,y} b_{yx}^{(\beta)}\eta(X, F_y^{(\beta)})E_x^{(\beta)}, F_z^{(\beta)}\right) &= \sum_{x,y} b_{yx}^{(\beta)}\eta(X, F_y^{(\beta)})\eta(E_x^{(\beta)}, F_z^{(\beta)}) \\ &= \sum_{y} \delta_{yz}\eta(X, F_y^{(\beta)}) = \eta(X, F_z^{(\beta)}), \end{aligned}$$

for all $1 \le z \le m$. (3.7) follows since $\eta_{|\mathfrak{u}^+(\chi)_\beta \times \mathfrak{u}^-(\chi)_{-\beta}}$ is non-degenerate. □

Let $\mathcal{C}_\beta$ be the canonical element of $\mathfrak{u}^+(\chi)_\beta \otimes \mathfrak{u}^-(\chi)_{-\beta}$, i.e.

$$\mathcal{C}_\beta = \sum_{x,y=1}^{m_\beta} b_{yx}^{(\beta)} E_x^{(\beta)} \otimes F_y^{(\beta)}.$$

Lemma 3.3. *Let $1 \le i \le \theta$. The following identities hold:*

$$[1 \otimes E_i, \mathcal{C}_{\beta+\alpha_i}] = \mathcal{C}_\beta(E_i \otimes L_i) - (E_i \otimes K_i)\mathcal{C}_\beta, \tag{3.9}$$

$$[\mathcal{C}_{\beta+\alpha_i}, F_i \otimes 1] = (L_i \otimes F_i)\mathcal{C}_\beta - \mathcal{C}_\beta(K_i \otimes F_i). \tag{3.10}$$

Proof. We prove (3.9). Let $Y \in \mathfrak{u}^-(\chi)_{-\beta-\alpha_i}$. Let Y', $Y'' \in \mathfrak{u}^-(\chi)_{-\beta}$ be such that $[E_i, Y] = Y'K_i - L_iY''$. Using (3.7), we have

$$
\begin{aligned}
(\hat{\eta}^{\geqq}_Y \otimes \mathrm{id})([1 \otimes E_i, \mathcal{C}_{\beta+\alpha_i}]) &= \sum_{x,y} b^{(\beta+\alpha_i)}_{yx} \eta(E^{(\beta+\alpha_i)}_x, Y)[E_i, F^{(-\beta-\alpha_i)}_y] \\
&= \left[E_i, \sum_{x,y} b^{(\beta+\alpha_i)}_{yx} \eta(E^{(\beta+\alpha_i)}_x, Y) F^{(-\beta-\alpha_i)}_y\right] = [E_i, Y]. \qquad (3.11)
\end{aligned}
$$

Now using (3.4), (3.7) and (3.11), we compute:

$$
\begin{aligned}
&(\hat{\eta}^{\geqq}_Y \otimes \mathrm{id})(\mathcal{C}_\beta(E_i \otimes L_i) - (E_i \otimes K_i)\mathcal{C}_\beta) \\
&\quad = \sum_{x,y} b^{(\beta)}_{yx} (\eta(E^{(\beta)}_x E_i, Y) F^{(\beta)}_y L_i - \eta(E_i E^{(\beta)}_x, Y) K_i F^{(\beta)}_y) \\
&\quad = \sum_{x,y} b^{(\beta)}_{yx} (\eta(E_i \otimes E^{(\beta)}_x, \Delta(Y)) F^{(\beta)}_y L_i - \eta(E^{(\beta)}_x \otimes E_i, \Delta(Y)) K_i F^{(\beta)}_y) \\
&\quad = \sum_{x,y} b^{(\beta)}_{yx} (-\eta(E^{(\beta)}_x, Y'') F^{(\beta)}_y L_i + \eta(E^{(\beta)}_x, Y') K_i F^{(\beta)}_y) \\
&\quad = -Y''L_i + K_iY' = [E_i, Y] = (\hat{\eta}^{\geqq}_Y \otimes \mathrm{id})([1 \otimes E_i, \mathcal{C}_{\beta+\alpha_i}]).
\end{aligned}
$$

Since $\eta_{|\mathfrak{u}^+(\chi)_\beta \times \mathfrak{u}^-(\chi)_{-\beta}}$ is non-degenerate, we have (3.9). Similarly we obtain (3.10). $\square$

Lemma 3.4. *Let $\mathcal{C}'_\beta := (K^\beta \otimes 1)(\mathcal{S} \otimes \mathrm{id})(\mathcal{C}_\beta)$. For every $\alpha \in \mathbb{N}_0^\theta$,*

$$
\sum_{\substack{\beta,\, \gamma \in \mathbb{N}_0^\theta \\ \beta+\gamma=\alpha}} \mathcal{C}_\beta \mathcal{C}'_\gamma = \delta_{\alpha,0} = \sum_{\substack{\beta,\, \gamma \in \mathbb{N}_0^\theta \\ \beta+\gamma=\alpha}} \mathcal{C}'_\beta \mathcal{C}_\gamma. \qquad (3.12)
$$

Proof. If $\alpha = 0$, (3.12) is clear. Assume $\alpha \neq 0$. We show the first equation of (3.12). Since $\eta_{|\mathfrak{u}^+(\chi)_\alpha \times \mathfrak{u}^-(\chi)_{-\alpha}}$ is non-degenerate, it suffices to show that

$$
\sum_{\substack{\beta,\, \gamma \in \mathbb{N}_0^\theta \\ \beta+\gamma=\alpha}} (\hat{\eta}^{\geqq}_Y \otimes \mathrm{id}_{\mathfrak{u}(\chi)})(\mathcal{C}_\beta \mathcal{C}'_\gamma) = 0, \qquad \text{for all } Y \in \mathfrak{u}^-(\chi)_{-\alpha}. \qquad (3.13)
$$

Write $\Delta(Y) = \sum_{\substack{\beta,\, \gamma \in \mathbb{N}_0^\theta \\ \beta+\gamma=\alpha}} Y^{(\beta,\gamma)}(1 \otimes L^\beta)$, where $Y^{(\beta,\gamma)} \in \mathfrak{u}^-(\chi)_{-\beta} \otimes \mathfrak{u}^-(\chi)_{-\gamma}$. Further write $Y^{(\beta,\gamma)} = \sum_m Y^{(\beta,\gamma)}_{-\beta,m} \otimes Y^{(\beta,\gamma)\prime}_{-\gamma,m}$, where $Y^{(\beta,\gamma)}_{-\beta,m} \in \mathfrak{u}^-(\chi)_{-\beta}$ and

$Y_{-\gamma,m}^{(\beta,\gamma)\prime} \in \mathfrak{u}^-(\chi)_{-\gamma}$. The left hand side of (3.13) is

$$\begin{aligned}
&\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_{\substack{x,x',\\ y,y'}} b_{yx}^{(\beta)} b_{y'x'}^{(\gamma)} \eta(E_x^{(\beta)} K^\gamma \mathcal{S}(E_{x'}^{(\gamma)}), Y) F_y^{(\beta)} F_{y'}^{(\gamma)}\\
&\quad= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)} b_{y'x'}^{(\gamma)} \eta(E_x^{(\beta)}, Y_{\gamma,m}^{(\beta,\gamma)\prime} L^\beta) \eta(K^\gamma \mathcal{S}(E_{x'}^{(\gamma)}), Y_{\beta,m}^{(\beta,\gamma)}) F_y^{(\beta)} F_{y'}^{(\gamma)}\\
&\quad= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)} b_{y'x'}^{(\gamma)} \eta(E_x^{(\beta)}, Y_{\gamma,m}^{(\beta,\gamma)\prime} L^\beta) \eta(\mathcal{S}(E_{x'}^{(\gamma)} K^\gamma), Y_{\beta,m}^{(\beta,\gamma)}) F_y^{(\beta)} F_{y'}^{(\gamma)}\\
&\quad= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)} b_{y'x'}^{(\gamma)} \eta(E_x^{(\beta)}, Y_{\gamma,m}^{(\beta,\gamma)\prime} L^\beta) \eta(E_{x'}^{(\gamma)} K^\gamma, \mathcal{S}^{-1}(Y_{\beta,m}^{(\beta,\gamma)})) F_y^{(\beta)} F_{y'}^{(\gamma)}\\
&\quad= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)} b_{y'x'}^{(\gamma)} \eta(E_x^{(\beta)}, Y_{\gamma,m}^{(\beta,\gamma)\prime} L^\beta) \eta(E_{x'}^{(\gamma)} K^\gamma, \mathcal{S}^{-1}(Y_{\beta,m}^{(\beta,\gamma)})) L^\gamma\\
&= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_{\substack{x,x',\\ y,y'}} b_{y'x'}^{(\gamma)} b_{yx}^{(\beta)} \eta(K^\gamma, L^\gamma) (\sum_m \eta(E_x^{(\beta)}, Y_{\gamma,m}^{(\beta,\gamma)\prime}) \eta(E_{x'}^{(\gamma)}, \mathcal{S}^{-1}(Y_{\beta,m}^{(\beta,\gamma)})) L^\gamma\\
&\quad= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_m \chi(\gamma,\gamma) Y_{\gamma,m}^{(\beta,\gamma)\prime} \mathcal{S}^{-1}(Y_{\beta,m}^{(\beta,\gamma)}) L^\gamma\\
&\quad= \sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} \sum_m Y_{\gamma,m}^{(\beta,\gamma)\prime} L^\gamma \mathcal{S}^{-1}(Y_{\beta,m}^{(\beta,\gamma)}) = \varepsilon(Y) = 0,
\end{aligned}$$

where we use (3.8) and the graduation of $\mathfrak{u}(\chi)$. The second equation of (3.12) is obtained in a similar way. □

Lemma 3.5. *The following identities hold:*

$$(\mathrm{id}\otimes\Delta)(\mathcal{C}_\alpha) = \sum_{\beta+\gamma=\alpha} \mathcal{C}_\beta^{(1,3)} \mathcal{C}_\gamma^{(1,2)} (1\otimes 1\otimes L^\gamma), \tag{3.14}$$

$$(\Delta\otimes\mathrm{id})(\mathcal{C}_\alpha) = \sum_{\beta+\gamma=\alpha} \mathcal{C}_\beta^{(1,3)} \mathcal{C}_\gamma^{(2,3)} (K^\gamma\otimes 1\otimes 1). \tag{3.15}$$

Proof. We show (3.14). Given $X_1 \in \mathfrak{u}^+(\chi)_\gamma$ and $X_2 \in \mathfrak{u}^+(\chi)_\beta$, we compute

$$\begin{aligned}
(\mathrm{id}\otimes\hat{\eta}_{X_1}^{\leq}\otimes\hat{\eta}_{X_2}^{\leq})(\mathrm{id}\otimes\Delta)(\mathcal{C}_\alpha) &= \sum_{x,y} b_{yx}^{(\alpha)} \eta(X_2 X_1, F_y^{(-\alpha)}) E_x^{(\alpha)}\\
= X_2 X_1 &= \sum_{x'',y'',x',y'} b_{y''x''}^{(\beta)} b_{y'x'}^{(\gamma)} \eta(X_2, F_{y''}^{(\beta)}) \eta(X_1, F_{y'}^{(-\gamma)}) E_{x''}^{(\beta)} E_{x'}^{(\gamma)}\\
&= (\mathrm{id}\otimes\hat{\eta}_{X_1}^{\leq}\otimes\hat{\eta}_{X_2}^{\leq})(\mathcal{C}_\beta^{(1,3)} \mathcal{C}_\gamma^{(1,2)}),
\end{aligned}$$

where we use (3.7) twice. Since

$$(\mathrm{id}\otimes\Delta)(\mathcal{C}_\alpha)\in\sum_{\beta+\gamma=\alpha}\mathfrak{u}^+(\chi)_\alpha\otimes\mathfrak{u}^+(\chi)_\gamma\otimes\mathfrak{u}^+(\chi)_\beta L^\gamma,$$

we prove that (3.14) holds. Similarly we obtain (3.15). □

3.2. *R*-**matrix for finite dimensional** $\mathfrak{u}(\chi)$**-modules.** Fix V_1, V_2, V_3 three finite dimensional $\mathfrak{u}(\chi)$-modules, with associated $\mathbf{k}$-algebra homomorphisms $\rho_x:\mathfrak{u}(\chi)\to\mathrm{End}_{\mathbf{k}}(V_x)$, $x\in\{1,2,3\}$, such that there exist an element $v_x\in V_x$ and a $\mathbf{k}$-algebra homomorphism $\Lambda_x:\mathfrak{u}^0(\chi)\to\mathbf{k}$ for each $x\in\{1,2,3\}$ satisfying

$$X\cdot v_x=\Lambda_x(X)v_x \text{ for all } X\in\mathfrak{u}^0(\chi), \qquad V_x=\mathfrak{u}^-(\chi)\cdot v_x,$$
$$E_i\cdot v_x=0 \text{ for all } 1\le i\le\theta.$$

If $\mathcal{F}=\sum_z\mathcal{F}'_z\otimes\mathcal{F}''_z\in\mathrm{End}_{\mathbf{k}}(V_x\otimes V_y)\cong\mathrm{End}_{\mathbf{k}}(V_x)\otimes\mathrm{End}_{\mathbf{k}}(V_y)$, $1\le x<y\le 3$, we set $\mathcal{F}^{(x,y)}\in\mathrm{End}_{\mathbf{k}}(V_1\otimes V_2\otimes V_3)\cong\mathrm{End}_{\mathbf{k}}(V_1)\otimes\mathrm{End}_{\mathbf{k}}(V_2)\otimes\mathrm{End}_{\mathbf{k}}(V_3)$ as

$$\mathcal{F}^{(x,y)}=\sum_z\mathcal{F}'_z\otimes\mathcal{F}''_z\otimes\mathrm{id}_{V_3} \qquad \text{if } x=1,y=2,$$
$$\mathcal{F}^{(x,y)}=\sum_z\mathcal{F}'_z\otimes\mathrm{id}_{V_2}\otimes\mathcal{F}''_z \qquad \text{if } x=1,y=3,$$
$$\mathcal{F}^{(x,y)}=\mathrm{id}_{V_1}\otimes\sum_z\mathcal{F}'_z\otimes\mathcal{F}''_z \qquad \text{if } x=2,y=3.$$

Now define $f_{xy}\in\mathrm{GL}_{\mathbf{k}}(V_x\otimes V_y)$ by

$$f_{xy}(Xv_x\otimes Yv_y):=\chi(\beta,\alpha)\Lambda_x(K^{-\beta})\Lambda_y(L^\alpha)Xv_x\otimes Yv_y$$

for α, $\beta\in\mathbb{N}_0^\theta$ and $X\in\mathfrak{u}^-(\chi)_{-\alpha}$, $Y\in\mathfrak{u}^-(\chi)_{-\beta}$. Set also

$$\mathcal{C}_{xy}:=\sum_{\beta\in\mathbb{N}_0^\theta}(\rho_x\otimes\rho_y)(\mathcal{C}_\beta), \qquad R_{xy}:=\mathcal{C}_{xy}f_{xy}^{-1}.$$

Lemma 3.6. *For each* $1\le i\le\theta$ *and* $\check{X}\in V_x\otimes V_y$,

$$f_{xy}((E_i\otimes 1)\check{X})=(E_i\otimes L_i^{-1})f_{xy}(\check{X}), \tag{3.16}$$
$$f_{xy}((1\otimes E_i)\check{X})=(K_i\otimes E_i)f_{xy}(\check{X}), \tag{3.17}$$
$$f_{xy}((F_i\otimes 1)\check{X})=(F_i\otimes L_i)f_{xy}(\check{X}), \tag{3.18}$$
$$f_{xy}((1\otimes F_i)\check{X})=(K_i^{-1}\otimes F_i)f_{xy}(\check{X}). \tag{3.19}$$

Proof. We show (3.16). For each $X\in\mathfrak{u}^-(\chi)_{-\beta}$, $Y\in\mathfrak{u}^-(\chi)_{-\gamma}$,

$$\begin{aligned} f_{xy}((E_i\otimes 1)Xv_x\otimes Yv_y) &= f_{xy}(E_iXv_x\otimes Yv_y)\\ &=\chi(\gamma,\beta-\alpha_i)\Lambda_x(K^{-\gamma})\Lambda_y(L^{\beta-\alpha_i})E_iXv_x\otimes Yv_y\\ &=(E_i\otimes L_i^{-1})f_{xy}(Xv_x\otimes Yv_y).\end{aligned}$$

Thus we have (3.16). Similarly we obtain (3.17), (3.18) and (3.19). □

Now we are ready to obtain the *R*-matrix for the modules V_x, $1\le x\le 3$.

Theorem 3.7. ***(i)*** *$\mathcal{C}_{xy} \in \mathrm{GL}_{\mathbf{k}}(V_x \otimes V_y)$ and*

$$\mathcal{C}_{xy}^{-1} = \sum_{\beta \in \mathbb{N}_0^\theta} (\rho_x \otimes \rho_y)(K^\beta \otimes 1)(\mathcal{S} \otimes \mathrm{id})(\mathcal{C}_\beta). \tag{3.20}$$

(ii) *For every $X \in \mathfrak{u}(\chi)$*

$$R_{xy}(\rho_x \otimes \rho_y)(\Delta(X))R_{xy}^{-1} = (\rho_x \otimes \rho_y)((\tau \circ \Delta)(X)). \tag{3.21}$$

(iii) *The following identities hold:*

$$\sum_{\beta \in \mathbb{N}_0^\theta} (\rho_1 \otimes \rho_2 \otimes \rho_3)((\Delta \otimes \mathrm{id}_{\mathfrak{u}(\chi)})(\mathcal{C}_\beta)) = \mathcal{C}_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}\mathcal{C}_{23}^{(2,3)}f_{13}^{(1,3)}, \tag{3.22}$$

$$\sum_{\beta \in \mathbb{N}_0^\theta} (\rho_1 \otimes \rho_2 \otimes \rho_3)((\mathrm{id}_{\mathfrak{u}(\chi)} \otimes \Delta)(\mathcal{C}_\beta)) = \mathcal{C}_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}\mathcal{C}_{12}^{(1,2)}f_{13}^{(1,3)}. \tag{3.23}$$

(iv) *The elements R_{xy} satisfy:*

$$R_{12}^{(1,2)}R_{13}^{(1,3)}R_{23}^{(2,3)} = R_{23}^{(2,3)}R_{13}^{(1,3)}R_{12}^{(1,2)}. \tag{3.24}$$

Proof. ***(i)*** This immediately follows from (3.12).

(ii) As we have algebra maps on both sides of the identity, it is enough to prove it for the generators of $\mathfrak{u}(\chi)$, and it follows by using Lemmata 3.3, 3.6. For example, for each $\check{X} \in V_x \otimes V_y$, by (3.16), (3.17), (3.9) we have

$$\begin{aligned}
(R_{xy}\Delta(E_i) - (\tau \circ \Delta)(E_i)R_{xy})\check{X} &= (\mathcal{C}_{xy}f_{xy}^{-1}\Delta(E_i) - (\tau \circ \Delta)(E_i)\mathcal{C}_{xy}f_{xy}^{-1})\check{X} \\
&= \sum_{\beta \in \mathbb{N}_0^\theta} (\mathcal{C}_\beta f_{xy}^{-1}(E_i \otimes 1 + K_i \otimes E_i) - (1 \otimes E_i + E_i \otimes K_i)\mathcal{C}_\beta f_{xy}^{-1})\check{X} \\
&= \sum_{\beta \in \mathbb{N}_0^\theta} (\mathcal{C}_\beta(E_i \otimes L_i + 1 \otimes E_i) - (1 \otimes E_i + E_i \otimes K_i)\mathcal{C}_\beta)f_{xy}^{-1}\check{X} \\
&= \sum_{\beta \in \mathbb{N}_0^\theta} ([1 \otimes E_i, \mathcal{C}_{\beta+\alpha_i}] - [1 \otimes E_i, \mathcal{C}_\beta])f_{xy}^{-1}\check{X} \\
&= -\sum_{\beta \in \mathbb{N}_0^\theta, \beta-\alpha_i \notin \mathbb{N}_0^\theta} [1 \otimes E_i, \mathcal{C}_\beta]f_{xy}^{-1}\check{X} = 0.
\end{aligned}$$

(iii) It can be proved by using Lemmata 3.5, 3.6. In fact, we compute for each $\check{X} \in V_x \otimes V_y$:

$$\begin{aligned}
\mathcal{C}_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}\mathcal{C}_{23}^{(2,3)}f_{13}^{(1,3)}\check{X} &= \sum_{\alpha,\, \gamma \in \mathbb{N}_0^\theta} \mathcal{C}_\alpha^{(1,3)}(f_{13}^{(1,3)})^{-1}(\mathcal{C}_\gamma^{(2,3)}f_{13}^{(1,3)}(\check{X})) \\
&= \sum_{\alpha,\, \gamma \in \mathbb{N}_0^\theta} \mathcal{C}_\alpha^{(1,3)}\mathcal{C}_\gamma^{(2,3)}(K^\gamma \otimes 1 \otimes 1)\check{X} \\
&= \sum_{\beta \in \mathbb{N}_0^\theta} (\rho_1 \otimes \rho_2 \otimes \rho_3)((\Delta \otimes \mathrm{id}_{\mathfrak{u}(\chi)})(\mathcal{C}_\beta))\check{X}.
\end{aligned}$$

(iv) In this case the proof follows by Lemma 3.3 and the previous claims:

$$\begin{aligned}
R_{12}^{(1,2)}R_{13}^{(1,3)}R_{23}^{(2,3)} &= R_{12}^{(1,2)}\mathcal{C}_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}\mathcal{C}_{23}^{(2,3)}(f_{23}^{(2,3)})^{-1} \\
&= \sum_{\beta\in\mathbb{N}_0^\theta} R_{12}^{(1,2)}(\rho_1\otimes\rho_2\otimes\rho_3)((\Delta\otimes\mathrm{id}_{\mathfrak{u}(\chi)})(\mathcal{C}_\beta))(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\
&= \sum_{\beta\in\mathbb{N}_0^\theta} (\rho_1\otimes\rho_2\otimes\rho_3)((\tau\circ\Delta)\otimes\mathrm{id}_{\mathfrak{u}(\chi)})(\mathcal{C}_\beta)R_{12}^{(1,2)}(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\
&= \mathcal{C}_{23}^{(2,3)}(f_{23}^{(2,3)})^{-1}\mathcal{C}_{13}^{(1,3)}f_{23}^{(2,3)}R_{12}^{(1,2)}(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\
&= R_{23}^{(2,3)}R_{13}^{(1,3)}f_{13}^{(1,3)}f_{23}^{(2,3)}R_{12}^{(1,2)}(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\
&= R_{23}^{(2,3)}R_{13}^{(1,3)}R_{12}^{(1,2)}.
\end{aligned}$$

□

4. R-matrices of quantum doubles of Nichols algebras with finite root systems

For this section we fix χ such that $M = |\Delta_+^\chi| < \infty$. First we recall a series of results from [HY2, Section 4], which will be useful to compute explicitly the universal R-matrix. Then we relate them with the chains of coideal subalgebras of [HS], and compute the desired R-matrices of quantum doubles of Nichols algebras with finite root systems. Finally we show some applications of the previous results to relate different PBW bases.

4.1. **PBW bases and Lusztig automorphisms.** Set an element $w = 1_\chi s_{i_1}s_{i_2}\cdots s_{i_M}$ of maximal length of $\mathcal{W}(\chi)$. Denote

$$\beta_k := s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad 1\le k\le M, \tag{4.1}$$

so $\beta_k \neq \beta_l$ if $k\neq l$, and $\Delta_+^\chi = \{\beta_k | 1\le k\le M\}$. Set $q_k := \chi(\beta_k,\beta_k)$, and N_k the order of q_k, which is possibly infinite. As in [HY2, Section 4], set

$$E_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}(E_{i_k}) \in \mathfrak{u}(\chi)^+_{\beta_k}, \quad \overline{E}_{\beta_k} = T^-_{i_1}\cdots T^-_{i_{k-1}}(E_{i_k}) \in \mathfrak{u}(\chi)^+_{\beta_k},$$
$$F_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}(F_{i_k}) \in \mathfrak{u}(\chi)^-_{\beta_k}, \quad \overline{F}_{\beta_k} = T^-_{i_1}\cdots T^-_{i_{k-1}}(F_{i_k}) \in \mathfrak{u}(\chi)^-_{\beta_k},$$

for $1\le k\le M$.

Theorem 4.1. [HY2, Theorems 4.5, 4.8, 4.9] *The sets*

$$\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_1}^{a_1} \,|0\le a_k<N_k,\, 1\le k\le M\},$$
$$\{\overline{E}_{\beta_M}^{a_M}\overline{E}_{\beta_{M-1}}^{a_{M-1}}\cdots \overline{E}_{\beta_1}^{a_1} \,|0\le a_k<N_k,\, 1\le k\le M\},$$

are bases of the vector space $\mathfrak{u}^+(\chi)$, and the sets

$$\{F_{\beta_M}^{a_M}F_{\beta_{M-1}}^{a_{M-1}}\cdots F_{\beta_1}^{a_1} \,|0\le a_k<N_k,\, 1\le k\le M\},$$
$$\{\overline{F}_{\beta_M}^{a_M}\overline{F}_{\beta_{M-1}}^{a_{M-1}}\cdots \overline{F}_{\beta_1}^{a_1} \,|0\le a_k<N_k,\, 1\le k\le M\},$$

are bases of the vector space $\mathfrak{u}^-(\chi)$. *Moreover, for each pair* $1 \le k < l \le M$,

$$\begin{aligned}
E_{\beta_k}E_{\beta_l} - \chi(\beta_k,\beta_l)E_{\beta_l}E_{\beta_k} &= \sum c_{a_{k+1},\dots,a_{l-1}} E_{\beta_{k+1}}^{a_{k+1}} \cdots E_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}^+(\chi),\\
\overline{E}_{\beta_k}\overline{E}_{\beta_l} - \chi^{-1}(\beta_k,\beta_l)\overline{E}_{\beta_l}\overline{E}_{\beta_k} &= \sum \overline{c}_{a_{k+1},\dots,a_{l-1}} E_{\beta_{k+1}}^{a_{k+1}} \cdots E_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}^+(\chi),\\
F_{\beta_k}F_{\beta_l} - \chi(\beta_k,\beta_l)F_{\beta_l}F_{\beta_k} &= \sum d_{a_{k+1},\dots,a_{l-1}} F_{\beta_{k+1}}^{a_{k+1}} \cdots F_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}^-(\chi),\\
\overline{F}_{\beta_k}\overline{F}_{\beta_l} - \chi^{-1}(\beta_k,\beta_l)\overline{F}_{\beta_l}\overline{F}_{\beta_k} &= \sum \overline{d}_{a_{k+1},\dots,a_{l-1}} F_{\beta_{k+1}}^{a_{k+1}} \cdots F_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}^+(\chi),
\end{aligned}$$

for some $c_{a_{k+1},\dots,a_{l-1}}, \overline{c}_{a_{k+1},\dots,a_{l-1}}, d_{a_{k+1},\dots,a_{l-1}}, \overline{d}_{a_{k+1},\dots,a_{l-1}} \in \mathbf{k}$. □

Note that $E_{\beta_k}E_{\beta_l} - \chi(\beta_k,\beta_l)E_{\beta_l}E_{\beta_k} = [E_{\beta_k}, E_{\beta_l}]_c$.

Now we want to describe the coproduct of the elements of these PBW generators. First we introduce the following subspaces of $\mathfrak{u}(\chi)$:

$$\begin{aligned}
B_+^l &:= \langle\{E_{\beta_l}^{a_l}E_{\beta_{l-1}}^{a_{l-1}}\cdots E_{\beta_1}^{a_1} \,|0 \le a_k < N_k\}\rangle \subseteq \mathfrak{u}^+(\chi),\\
C_+^l &:= \langle\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_1}^{a_1} \,|\exists j > l \text{ s.t. } a_j \ne 0\}\rangle \subseteq \mathfrak{u}^+(\chi),\\
D_+^l &:= \langle\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_1}^{a_1} \,|\exists j < l \text{ s.t. } a_j \ne 0\}\rangle \subseteq \mathfrak{u}^+(\chi),\\
B_-^l &:= \langle\{F_{\beta_l}^{a_l}F_{\beta_{l-1}}^{a_{l-1}}\cdots F_{\beta_1}^{a_1} \,|0 \le a_k < N_k\}\rangle \subseteq \mathfrak{u}^-(\chi),\\
C_-^l &:= \langle\{F_{\beta_M}^{a_M}F_{\beta_{M-1}}^{a_{M-1}}\cdots F_{\beta_1}^{a_1} \,|\exists j > l \text{ s.t. } a_j \ne 0\}\rangle \subseteq \mathfrak{u}^-(\chi),\\
D_-^l &:= \langle\{F_{\beta_M}^{a_M}F_{\beta_{M-1}}^{a_{M-1}}\cdots F_{\beta_1}^{a_1} \,|\exists j < l \text{ s.t. } a_j \ne 0\}\rangle \subseteq \mathfrak{u}^-(\chi),
\end{aligned}$$

$1 \le l \le M$; $\langle S\rangle$ denotes the subspace spanned by a subset S of $\mathfrak{u}(\chi)$.

Proposition 4.2. B_+^l *(respectively,* B_-^l*) is a right (respectively, left) coideal subalgebra of* $\mathfrak{u}^+(\chi)$ *(respectively,* $\mathfrak{u}^-(\chi)$*).*

Proof. For each $1 \le l \le M$, set $w_l = 1_\chi s_{i_1}s_{i_{M-1}}\cdots s_{i_l}$, and the corresponding right coideal subalgebra $B(w_l)$ of $\mathfrak{u}^+(\chi)$ (for the braided coproduct $\underline{\Delta}$) as in Theorem 2.5; then its Hilbert series is

$$\mathcal{H}_{B(w_l)} = \prod_{j=1}^{l} \mathbf{q}_{N_l}(X^{\beta_l}).$$

By the definition of $B(w_l)$ in [HS] (which involves the T_j's) it follows that $E_{\beta_j} \in B(w_l)$ for each $1 \le j \le k$. Therefore $B_+^l \subseteq B(w_l)$, because $B(w_l)$ is a subalgebra. But both N_0^θ-graded vector subspaces of $\mathfrak{u}^+(\chi)$ have the same Hilbert series by Theorem 4.1, so $B_+^l = B(w_l)$ is a right coideal subalgebra.

The statement about B_-^l is analogous because $\mathfrak{u}^-(\chi) \simeq \mathcal{B}(V^*)^{\mathrm{cop}}$. □

Corollary 4.3. *For each* $1 \le l \le M$,

$$\begin{aligned}
\underline{\Delta}(E_{\beta_l}) &\in E_{\beta_l} \otimes 1 + 1 \otimes E_{\beta_l} + B_+^{l-1} \otimes C_+^l,\\
\underline{\Delta}(F_{\beta_l}) &\in F_{\beta_l} \otimes 1 + 1 \otimes F_{\beta_l} + C_+^l \otimes B_-^{l-1}.
\end{aligned}$$

Proof. By the previous Proposition and the fact that $\mathfrak{u}^+(\chi)$ is a graded connected Hopf algebra,

$$\underline{\Delta}(E_{\beta_l}) = E_{\beta_l} \otimes 1 + 1 \otimes E_{\beta_l} + \sum E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} \otimes X_{a_1,\dots,a_{l-1}},$$

for some $X_{a_1,\dots,a_{l-1}} \in \mathfrak{u}^+(\chi)$. Now express these elements in terms of the PBW basis:

$$X_{a_1,\dots,a_{l-1}} = \sum c_{b_m,\dots,b_1}^{a_{l-1},\dots,a_1} E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_1}^{b_1}.$$

Suppose that $c_{b_m,\dots,b_1}^{a_{l-1},\dots,a_1} \neq 0$. As $\mathfrak{u}^+(\chi)$ is $\mathbb{N}_0^\theta$-graded, $\beta_l = \sum_{b_i \neq 0} b_i \beta_i + \sum_{a_j \neq 0} a_j \beta_j$. As j runs between 1 and $l-1$, Theorem 2.4 implies that there exists $i > l$ such that $b_i \neq 0$.

The proof for F_{β_l} is analogous. □

More generally, we can describe the coproduct of each PBW generator. In this case we can only describe the left hand side of the tensor product.

Proposition 4.4. *For each* $1 \leq l \leq M$, $1 \leq a_l < N_l$,

$$\begin{aligned}\underline{\Delta}(E_{\beta_l}^{a_l} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}) \in & \sum_{p=0}^{a_l} \binom{a_l}{p}_{q_l} E_{\beta_l}^{p} \otimes E_{\beta_l}^{a_l - p} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} \\ & + E_{\beta_l}^{a_l} \cdots E_{\beta_1}^{a_1} \otimes 1 + (D_+^l \cap B_+^l) \otimes \mathfrak{u}^+(\chi), \\ \underline{\Delta}(F_{\beta_l}^{a_l} F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1}) \in & \sum_{p=0}^{a_l} \binom{a_l}{p}_{q_l} F_{\beta_l}^{a_l - p} F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} \otimes F_{\beta_l}^{p} \\ & + 1 \otimes F_{\beta_l}^{a_l} \cdots F_{\beta_1}^{a_1} + \mathfrak{u}^-(\chi) \otimes (D_-^l \cap B_-^l),\end{aligned}$$

Proof. We prove the statement for the E_{β_k}'s by induction on l; the proof for the F_{β_k}'s is analogous. The case $l = 1$ is trivial, because $E_{\beta_1} = E_{i_1}$ is primitive, so

$$\underline{\Delta}(E_{\beta_1}^{a_1}) = \sum_{p=0}^{a_1} \binom{a_1}{p}_{q_1} E_{\beta_1}^{p} \otimes E_{\beta_1}^{a_1 - p}.$$

Assume that it holds for $k < l$. Now we use induction on a_l. If $a_l = 1$,

$$\underline{\Delta}(E_{\beta_l} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}) = \underline{\Delta}(E_{\beta_l}) \underline{\Delta}(E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}).$$

Therefore we use inductive hypothesis, Corollary 4.3 and the fact that B_{l-1} is a subalgebra to conclude the proof. The inductive step on a_l is completely analogous, and close to the proof of results involving the coproduct of hyperletters in [Kh]. □

4.2. **Explicit computation of the universal R-matrix.** We will obtain now an explicit formula for the universal R- matrix when the Nichols algebra is finite-dimensional. By (2.1) it is enough to compute bases of $\mathfrak{u}^{\geq 0}(\chi)$ and $\mathfrak{u}^{\leq 0}(\chi)$, which are dual for η. Such bases will be those of Theorem 4.1.

The proof is similar to the one of [A2, Proposition 4.2], see also [Ro2].

Remark 4.5. Set for each $\alpha = (a_1, \dots, a_\theta) \in \mathbb{Z}^\theta$

$$K^\alpha := K_1^{a_1} \cdots K_\theta^{a_\theta} \in \mathfrak{u}^{+0}(\chi), \qquad L^\alpha := L_1^{a_1} \cdots L_\theta^{a_\theta} \in \mathfrak{u}^{-0}(\chi).$$

For each $\mathbb{Z}^\theta$-homogeneous element $E \in \mathfrak{u}(\chi)$ let $|E| \in \mathbb{Z}^\theta$ be its degree. Therefore,

$$\Delta(E) = E_{(1)} K^{|E_{(2)}|} \otimes E_{(2)}, \tag{4.2}$$

for each homogeneous $E \in \mathfrak{u}^+(\chi)$, where $\underline{\Delta}(E) = E_{(1)} \otimes E_{(2)}$ is the Sweedler notation for the braided comultiplication. Analogously, for each homogeneous $F \in \mathfrak{u}^-(\chi)$,

$$\Delta(F) = F_{(1)} \otimes F_{(2)} L^{|F_{(1)}|}. \tag{4.3}$$

Proposition 4.6. *Let $0 \le a_i, b_i \le N_i$, for each $1 \le i \le M$. Then*

$$\eta\left(E_{\beta_M}^{a_M} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_1}^{a_1}, F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_1}^{b_1}\right) = \prod_{i=1}^{M} \delta_{a_i,b_i} (a_i)_{q_i}! \eta_i^{a_i}, \tag{4.4}$$

where $\eta_i := \eta(E_{\beta_i}, F_{\beta_i})$ is not zero for all i.

Proof. We will prove (4.4) by induction on $\sum a_i$, $\sum b_i$; therefore $\eta_i \neq 0$ for all i because η is a non-degenerate pairing. It is clear if $\sum a_i = 0$. If $\sum a_i = 1$, then the PBW generator is just E_{β_j} for some j. For this case we apply decreasing induction on j. Note that $\eta\left(E_{\beta_j}, F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_1}^{b_1}\right) = 0$ when $\beta_j \neq \sum_l b_l \beta_l$, by Proposition 2.6. If $\beta_j = \sum_l b_l \beta_l$ and β_j is a simple root, the unique possibility is $b_j = 1$ and $b_l = 0$ for $l \neq j$. If β_j is not a simple root, then either $b_j = 1$ and $b_l = 0$ for $l \neq j$, or there exists $k > j$ such that $b_k > 0$ because the order is strongly convex. In the last case,

$$\begin{aligned}\eta\left(E_{\beta_j}, F_{\beta_k}^{b_k} F_{\beta_{k-1}}^{b_{k-1}} \cdots F_{\beta_1}^{b_1}\right) = \eta &\left((E_{\beta_j})_{(1)} K^{|(E_{\beta_j})_{(2)}|}, F_{\beta_k}\right) \\ &\eta\left((E_{\beta_j})_{(2)}, F_{\beta_k}^{b_k-1} F_{\beta_{k-1}}^{b_{k-1}} \cdots F_{\beta_1}^{b_1}\right) = 0,\end{aligned}$$

because $\eta\left((E_{\beta_j})_{(1)} K^{|(E_{\beta_j})_{(2)}|}, F_{\beta_k}\right) = 0$ by Corollary 4.3 and the inductive hypothesis.

Assume that $\sum a_i, \sum b_i > 0$ and we have proved the formula for sums smaller than these two. Set $k = \max\{i : a_i \neq 0\}$, $l = \max\{j : b_j \neq 0\}$, and suppose that $k \le l$ (otherwise the proof is analogous). By Proposition 4.4,

$$\begin{aligned}&\eta\left(E_{\beta_k}^{a_k} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1}, F_{\beta_l}^{b_l} F_{\beta_{l-1}}^{b_{l-1}} \cdots F_{\beta_1}^{b_1}\right) \\ =&\eta\left((E_{\beta_k}^{a_k} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1})_{(1)} K^{|(E_{\beta_k}^{a_k} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1})_{(2)}|}, F_{\beta_l}\right) \\ &\quad \eta\left((E_{\beta_1}^{a_1} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_k}^{a_k})_{(2)}, F_{\beta_l}^{b_l-1} F_{\beta_{l-1}}^{b_{l-1}} \cdots F_{\beta_1}^{b_1}\right) \\ =&(b_l)_{q_l} \eta_l \delta_{l,k} \eta\left(E_{\beta_k}^{a_k-1} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1}, F_{\beta_l}^{b_l-1} F_{\beta_{l-1}}^{b_{l-1}} \cdots F_{\beta_1}^{b_1}\right),\end{aligned}$$

so the proof follows by inductive hypothesis. ∎

Now we obtain a formula for the scalars η_i. The algebras $\mathfrak{u}^{\geq 0}(\chi)$, $\mathfrak{u}^{\leq 0}(\chi)$ are canonically $\mathbb{N}_0$-graded; we denote by $d(X)$, $d(Y)$ the degree of the homogeneous elements $X \in \mathfrak{u}^{\geq 0}(\chi)$, $Y \in \mathfrak{u}^{\leq 0}(\chi)$. In fact, if $X \in \mathfrak{u}^{\geq 0}(\chi)_\beta$, $Y \in \mathfrak{u}^{\leq 0}(\chi)_{-\beta}$, $\beta = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{N}_0^\theta$, then $d(X) = d(Y) = \sum_{i=1}^{\theta} n_i$.

Lemma 4.7. *$\eta_k = (-1)^{d(E_{\beta_k})}$ for all $1 \leq k \leq M$.*

Proof. By induction on k, it is easy to prove that

$$E_{\beta_k}F_{\beta_k} - F_{\beta_k}E_{\beta_k} = K^{\beta_k} - L^{\beta_k}. \tag{4.5}$$

On the other hand, by (3.2) we have that

$$E_{\beta_k}F_{\beta_k} = \eta\left((E_{\beta_k})_1, (F_{\beta_k})_1\right)\eta\left((E_{\beta_k})_3, \mathcal{S}((F_{\beta_k})_3)\right)(F_{\beta_k})_2(E_{\beta_k})_2. \tag{4.6}$$

Using (4.2) and the fact that $\mathfrak{u}^{\geq 0}(\chi)$ is $\mathbb{N}_0^\theta$-graded, we deduce that the unique term in $\Delta^{(2)}(E_{\beta_k})$ where appears K^{β_k} in the middle is $K^{\beta_k} \otimes K^{\beta_k} \otimes E_{\beta_k}$. If we want to compute the coefficient of in (4.6), it is enough to look at the term $1 \otimes 1 \otimes F_{\beta_k}$ in $\Delta^{(2)}(F_{\beta_k})$, because the components of different degree are orthogonal for η. Using the antipode axiom and that $\mathfrak{u}^{\leq 0}(\chi)$ is graded, we have that $\mathcal{S}(F_{\beta_k})$ is written as $(-1)^{d(F_{\beta_k})}F_{\beta_k}L^{-\beta_k}$ plus terms of lower degree. Then the coefficient of K^{β_k} in the right hand side of (4.6) is $(-1)^{d(F_{\beta_k})}\eta_k$, using again the orthogonality of the components of different degree. $\square$

We recall a generalization of Proposition 2.6. The main objective is to consider bosonizations of Nichols algebras by abelian groups, not only free abelian groups, and their quantum doubles. Similar generalizations can be found in [ARS, RaS], and also in [B] for finite groups.

Set a bicharacter $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$, and two abelian groups Γ, Λ. Assume that there exists elements $g_i \in \Gamma$, $\gamma_j \in \widehat{\Gamma}$ such that $\gamma_j(g_i) = q_{ij}$, and elements $h_i \in \Lambda$, $\lambda_j \in \widehat{\Lambda}$ such that $\lambda_j(h_i) = q_{ji}$. Assume that there exists a bicharacter $\mu : \Gamma \times \Lambda \to \mathbf{k}^\times$, such that $\mu(g_i, h_j) = q_{ij}$. Note that this happens for example when $\Gamma = \Lambda = \mathbb{Z}^\theta$, as in [H2, Section 4].

Set $V \in^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma} \mathcal{YD}$ as the vector space with a fixed basis $E_1, \dots, E_\theta$ such that $E_i \in V_{g_i}^{\gamma_i}$, $W \in^{\mathbf{k}\Lambda}_{\mathbf{k}\Lambda} \mathcal{YD}$ to the vector space with a fixed basis $F_1, \dots, F_\theta$ such that $F_i \in V_{h_i}^{\lambda_i}$. Let $\mathcal{B} = \mathcal{B}(V)\#\mathbf{k}\Gamma$ and $\mathcal{B}' = (\mathcal{B}(W)\#\mathbf{k}\Lambda)^{\mathrm{cop}}$.

Theorem 4.8. *There exists a unique skew-Hopf pairing $\eta : \mathcal{B} \otimes \mathcal{B}' \to \mathbf{k}$ such that for all $1 \leq i, j \leq \theta$ and all $g \in \Gamma$, $h \in \Lambda$,*

$$\eta(g, h) = \mu(g, h), \qquad \eta(E_i, F_j) = -\delta_{ij}, \qquad \eta(E_i, h) = \eta(g, F_j) = 0. \tag{4.7}$$

It satisfies the following condition: for all $E \in \mathfrak{u}^+(\chi)$, $F \in \mathfrak{u}^-(\chi)$, $g \in \mathcal{B}$, $h \in \mathcal{B}'$,

$$\eta(Eg, Fh) = \eta(E, F)\mu(g, h). \tag{4.8}$$

The restriction of η to $\mathcal{B}(V) \otimes \mathcal{B}(W)$ coincides with the one of the pairing in Proposition 2.6. $\square$

We work with the case $\Lambda = \widehat{\Gamma}$, Γ a finite group, μ the evaluation bicharacter, and $h_i = \gamma_i$, $\lambda_i = g_i$ under the canonical identification of the characters of $\widehat{\Gamma}$ with Γ. In this case η is non-degenerate. Call $\mathfrak{u}(\chi)$ to the Hopf algebra corresponding to this skew-Hopf pairing, following Subsection 2.1, and denote $\mathcal{B} = \mathfrak{u}^{\geq 0}(\chi)$, $\mathcal{B}' = \mathfrak{u}^{\leq 0}(\chi)$ by analogy with the previous sections. Two dual bases for $\eta|_{\mathbf{k}\Gamma\otimes\mathbf{k}\widehat{\Gamma}}$ are $\{g\}_{g\in\Gamma}$, $\{\delta_g\}_{g\in\Gamma}$, where $\delta_g = |\Gamma|^{-1}\sum_{\gamma\in\widehat{\Gamma}}\gamma(g^{-1})\,\gamma$. Therefore it has an R-matrix of the form:

$$\mathcal{R}_1 := \sum_{g\in\Gamma}\delta_g\otimes g = \frac{1}{|\Gamma|}\sum_{g\in\Gamma,\gamma\in\widetilde{\Gamma}}\gamma(g^{-1})\,\gamma\otimes g. \tag{4.9}$$

Theorem 4.9. *The universal R-matrix of $\mathfrak{u}(\chi)$ is given by the formula*

$$\mathcal{R} = \left(\prod \exp_{q_j}\left((-1)^{d(F_{\beta_k})}F_{\beta_j}\otimes E_{\beta_j}\right)\right)\mathcal{R}_1, \tag{4.10}$$

where the product is written in decreasing order.

Proof. By Proposition 4.6 and Theorem 4.8, the sets

$$\{E_{\beta_M}^{a_M}\cdots E_{\beta_1}^{a_1}g : 0\leq a_i<N_i, g\in\Gamma\},$$
$$\left\{\left(\prod_{i=1}^{M}(a_i)_{q_i}!\eta_i^{a_i}\right)^{-1} F_{\beta_M}^{b_M}\cdots F_{\beta_1}^{b_1}\delta_g : 0\leq b_i<N_i, g\in\Gamma\right\}$$

are bases of $\mathfrak{u}^{\geq 0}(\chi)$, $\mathfrak{u}^{\leq 0}(\chi)$, respectively, which are dual for η. As in Subsection 2.1, a formula for the R-matrix is given by:

$$\begin{aligned}\mathcal{R} &= \sum_{g\in\Gamma}\sum_{0\leq a_i<N_i}\left(\prod_{i=1}^{M}(a_i)_{q_i}!\eta_i^{a_i}\right)^{-1} F_{\beta_M}^{b_M}\cdots F_{\beta_1}^{b_1}\delta_g\otimes E_{\beta_M}^{a_M}\cdots E_{\beta_1}^{a_1}g\\ &= \left(\prod\left(\sum_{i=0}^{N_j-1}\frac{\eta_j^i}{(i)_{q_j}!}F_{\beta_j}^i\otimes E_{\beta_j}^i\right)\right)\left(\sum_{g\in\Gamma}\delta_g\otimes g\right),\end{aligned}$$

which ends the proof. □

4.3. **Further computations on convex PBW bases.** We can refine the coproduct expression of each E_β. In consequence we can obtain a family of left coideal subalgebras, induced by products of the same PBW generators. For each $1\leq l\leq M$, let

$$\mathbf{B}_+^l := \langle\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_l}^{a_l}\,|0\leq a_k<N_k\}\rangle\subseteq\mathfrak{u}^+(\chi),$$
$$\mathbf{B}_-^l := \langle\{F_{\beta_M}^{a_M}F_{\beta_{M-1}}^{a_{M-1}}\cdots F_{\beta_l}^{a_l}\,|0\leq a_k<N_k\}\rangle\subseteq\mathfrak{u}^-(\chi).$$

Lemma 4.10. *For each $1\leq l\leq M$,*

$$\underline{\Delta}(E_{\beta_l})\in E_{\beta_l}\otimes 1+1\otimes E_{\beta_l}+B_+^{l-1}\otimes\mathbf{B}_+^{l-1},$$
$$\underline{\Delta}(F_{\beta_l})\in F_{\beta_l}\otimes 1+1\otimes F_{\beta_l}+\mathbf{B}_-^{l-1}\otimes B_-^{l-1}.$$

Proof. Write both sides of $\underline{\Delta}(E_{\beta_l})$ as linear combinations of the elements of the PBW basis, and take a term

$$E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} \otimes E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_k}^{b_k}$$

appearing with non-zero coefficient c, where k is such that $b_k \neq 0$. Using the orthogonality of the elements of the PBW basis,

$$\begin{aligned} 0 \neq c\,\eta(E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} K^{|E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_k}^{b_k}|}, F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1}) \\ \eta(E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_k}^{b_k}, F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_k}^{b_k}) \\ = \eta(E_{\beta_l}, F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_k}^{b_k}). \end{aligned}$$

Suppose that $k < l$. Using last part of Theorem 4.1 repeatedly we see that

$$z := F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_k}^{b_k} \in D_-^l,$$

so $\eta(E_{\beta_l}, z) = 0$, a contradiction. Then $k \geq l$, and we end the proof. □

Proposition 4.11. $\mathbf{B}_+^l$ *(respectively,* $\mathbf{B}_-^l$*) is a left (respectively, right) coideal subalgebra of* $\mathfrak{u}^+(\chi)$ *(respectively,* $\mathfrak{u}^-(\chi)$*).*

Proof. It is a consequence of Lemma 4.10 and last part of Theorem 4.1. □

For the last part of this section we prove a result generalizing [Ro2, Theorem 22]. It establishes the uniqueness (up to scalars) of a PBW basis determining a filtration of coideal subalgebras, and it is useful to compare PBW bases coming from Lusztig isomorphisms as in the previous results, and PBW bases from combinatorics as [Kh]. Note that the first kind of PBW bases gives right and left coideal subalgebras, while some examples of the second family give left coideal subalgebras, see [A2, Section 3.3].

Theorem 4.12. *Let* $(\mathbf{E}_\beta)_{\beta \in \Delta_+^\chi}$ *be non-zero elements of* $\mathfrak{u}^+(\chi)$*, such that* $\mathbf{E}_\beta \in \mathfrak{u}^+(\chi)_\beta$*, and there exists an order* $\beta_M > \ldots > \beta_1$ *on the roots such that, for each* $1 \leq k \leq M$*, the elements* $\mathbf{E}_{\beta_M}^{a_M} \cdots \mathbf{E}_{\beta_k}^{a_k}$*,* $0 \leq a_j < N_{\beta_k}$*, determine a basis of a subspace* $\mathbf{Y}_k$*, which is a left coideal subalgebra of* $\mathfrak{u}^+(\chi)$*. Then the order on the roots is convex.*

Moreover, if $(E_\beta)_{\beta \in \Delta_+^\chi}$ *denote PBW generators for the corresponding expression of the element of maximal length of* $\mathcal{W}$*, then there exists non-zero scalars* c_β *such that* $\mathbf{E}_\beta = c_\beta\, E_\beta$*.*

Proof. The convexity on the order follows from the fact that the chain of coideal subalgebras $\mathbf{Y}_M \subsetneq \cdots \subsetneq \mathbf{Y}_1 = \mathcal{B}(V)$ coincides with $\mathbf{B}_+^M \subsetneq \cdots \subsetneq \mathbf{B}_+^1 = \mathcal{B}(V)$. The proof of this fact is exactly as in [A2, Theorem 3.16]. That is, $\mathbf{Y}_k = \mathbf{B}_+^k$ for all $1 \leq k \leq M$..

For the second statement, write $\mathbf{E}_{\beta_k} = \sum c(a_1, \cdots, a_M)\, E_{\beta_M}^{a_M} \cdots E_{\beta_1}^{a_1}$. If $c(a_1, \cdots, a_M) \neq 0$, then $\beta_k = \sum_j a_j \beta_j$, so $a_k = 1$, $a_j = 0$ for all $j \neq k$, or there exists $j < k$ such that $a_j \neq 0$. The second case is not possible because $\mathbf{E}_{\beta_k} \in \mathbf{Y}_k = \mathbf{B}_+^k$. Therefore, $\mathbf{E}_{\beta_k} = c_{\beta_k}\, E_{\beta_k}$ for some $c_{\beta_k} \in \mathbf{k}^\times$. □

Example 4.13. Let $(q_{ij})_{1\le i,j\le 2}$ be a matrix whose generalized Dynkin diagram is $\overset{\zeta}{\circ}\overset{\zeta^2}{\text{———}}\overset{-1}{\circ}$, ζ a root of unity of order 5, and χ the associated bicharacter on $\mathbb{Z}^2$. The element of maximal length on its Weyl groupoid has a reduced expression $w_0 = \mathrm{id}^\chi s_1s_2s_1s_2s_1s_2s_1s_2$. Then

$$\begin{aligned}\alpha_1 &< 3\alpha_1+\alpha_2 < 2\alpha_1+\alpha_2 < 5\alpha_1+3\alpha_2\\ &< 3\alpha_1+2\alpha_2 < 4\alpha_1+3\alpha_2 < \alpha_1+\alpha_2 < \alpha_2\end{aligned}$$

is the corresponding order on the roots. We obtain a PBW basis with generators E_β, $\beta\in\Delta_+^\chi$, using the Lusztig isomorphisms.

On the other hand we obtain a PBW basis of hyperletters $\mathbf{E}_\beta = [\ell_\beta]_c$, $\beta\in\Delta_+^\chi$, associated to Lyndon words ℓ_β as in [Kh]. We compute easily the corresponding Lyndon words using [A2, Corollary 3.17]:

$$\begin{aligned} \ell_{\alpha_1} &= x_1, & \ell_{3\alpha_1+\alpha_2} &= x_1^3x_2, & \ell_{4\alpha_1+3\alpha_2} &= x_1^2x_2x_1x_2x_2x_1x_2,\\ \ell_{\alpha_2} &= x_2, & \ell_{2\alpha_1+\alpha_2} &= x_1^2x_2, & \ell_{5\alpha_1+3\alpha_2} &= x_1^2x_2x_1^2x_2x_1x_2,\\ \ell_{\alpha_1+\alpha_2} &= x_1x_2, & \ell_{3\alpha_1+2\alpha_2} &= x_1^2x_2x_1x_2. \end{aligned}$$

We compute using the Shirshov decomposition, see [A2, Kh] and the references there in,

$$\begin{aligned} \mathbf{E}_{\alpha_1} &= x_1, & \mathbf{E}_{3\alpha_1+\alpha_2} &= (\mathrm{ad}_c\, x_1)^3x_2,\\ \mathbf{E}_{\alpha_2} &= x_2, & \mathbf{E}_{3\alpha_1+2\alpha_2} &= [\mathbf{E}_{2\alpha_1+\alpha_2}, \mathbf{E}_{\alpha_1+\alpha_2}]_c,\\ \mathbf{E}_{\alpha_1+\alpha_2} &= (\mathrm{ad}_c\, x_1)x_2, & \mathbf{E}_{4\alpha_1+3\alpha_2} &= [\mathbf{E}_{3\alpha_1+2\alpha_2}, \mathbf{E}_{\alpha_1+\alpha_2}]_c,\\ \mathbf{E}_{2\alpha_1+\alpha_2} &= (\mathrm{ad}_c\, x_1)^2x_2, & \mathbf{E}_{5\alpha_1+3\alpha_2} &= [\mathbf{E}_{2\alpha_1+\alpha_2}, \mathbf{E}_{3\alpha_1+2\alpha_2}]_c. \end{aligned}$$

By the previous theorem, there exists $c_\beta\in\mathbf{k}^\times$ such that $\mathbf{E}_\beta = c_\beta\, E_\beta$. It can be computed as the inverse of the coefficient of ℓ_β in E_β, because ℓ_β appears with coefficient 1 in $\mathbf{E}_\beta$.

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